A Simple Recursive Formula for Calculating the S-Parameters of Finite Periodic Structures

Cen Yen Ong, *Student Member, IEEE,* and Michal Okoniewski, *Senior Member, IEEE*

***Abstract*—A simple and efficient recursive formula to calculate the S-parameters of large, finite periodic structures is introduced. The technique utilizes measured or computed S matrices of structures with a small number of periods. The formula is verified on a simple rectangular waveguide with periodic gratings. Results from the formula are plotted against similar results obtained from finite-difference time-domain (FDTD) simulations, and they show excellent agreement. This formula can be used to compute the S-parameters of large microwave or optical periodic structures, which, by virtue of their finiteness may not be open to the application of periodic boundary conditions.**

***Index Terms*—Electromagnetic propagation, FDTD, filters, periodic structures.**

I. Introduction

PERIODIC structures are ubiquitous in the microwave and optical domain. The periodic perturbations, with dimensions on the order of the wavelength, significantly affect the propagation of electromagnetic waves, allowing the transmission of some, while reflecting or attenuating others. The frequency selectivity of periodic structures has thus been exploited in microwave and optical resonators and filters. Periodic structures are also integrated in distributed feedback lasers, distributed Bragg reflection lasers and quasiphase-matched second harmonic generators [1].

Electromagnetic propagation characteristics in periodic structures have been analyzed via the finite-difference time-domain (FDTD) method, the method of moments, the method of lines [2], the coupled-wave method [3], and by utilizing the Floquet–Bloch theorem [4]. Hybrid frequency-time or time-spatial techniques of analysis have also been explored [1], [5]. In most techniques, the analysis is simplified by the assumption of infinite periodicity, arguing that it offers a good model for the central elements of a large, but finite periodic element [6]. When infinite periodicity is assumed, the analysis of a large structure reduces to the solution of a single cell. In FDTD, periodic boundary conditions (PBCs) are used to emulate infinite periodicity at the walls of the unit cell. PBCs, however, are difficult to implement when the angle of incidence of the electromagnetic wave is all but normal [6], [7]. Moreover, the realistic modeling of finitely periodic structures is of major interest since the symmetry of infinite periodic structures often results in oversimplified electromagnetic properties [3].

Manuscript received November 28, 2001; revised April 22, 2002. This work was supported in part by the NSERC, Alberta Heritage Foundation for Science and Engineering Research and the Telecommunications Research Laboratories (TRLabs), Canada. The review of this letter was arranged by Associate Editor Dr. Rüdiger Vahldieck.

The authors are with the Department of Electrical and Computer Engineering, University of Calgary, Calgary, AB T2N 1N4 Canada (e-mail: ongc@enel.ucalgary.ca).

Publisher Item Identifier 10.1109/LMWC.2002.801140.

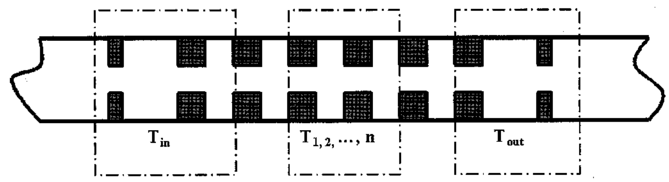


Fig. 1. Top view of periodic structure.

In a finite periodic structure, increasing the number of periods increases the resonance of the structure until the limit of saturation is reached [4]. As the number of periods increases, the computational problem increases in both memory and time. Performing FDTD simulations of large periodic structures is computationally intensive and, thus, prohibitively expensive.

On the other hand, it is relatively simple to measure or simulate structures with a relatively small number of periods. Consequently, a simple recursive formula to obtain the S-parameters of large finite periodic structures, based on the knowledge of S-parameters of smaller structures, is introduced. The formula abstracts from higher order modes that are generated at the periodic discontinuities within the structure by implicitly encapsulating their effect in the transmission of modes that exist at the input and output ports. This method thus applies to any general periodic structures with multi- or single-mode propagation.

II. Theory

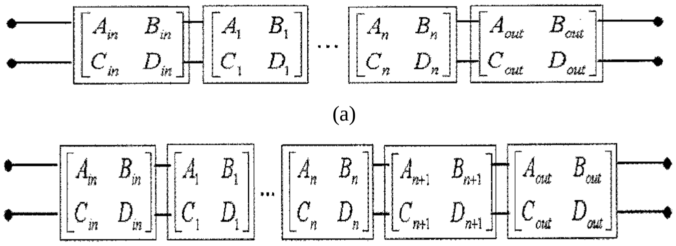
The S-parameters of an arbitrary structure can be measured or computed using previously mentioned modern electromagnetic analysis techniques. The $ABCD$ matrix of the same structure can then be calculated from the S-parameters using basic transformation equations [8].

Consider a simple structure that is periodic in one dimension, as illustrated in Fig. 1.

The structure can be mathematically modeled by cascading its composite transmission matrices, as in Fig. 2.

Let $\mathbf{T_n} = T_{in}T_1T_2\cdots T_nT_{out}$ where, $T_{in} = \begin{bmatrix} A_{in} & B_{in} \\ C_{in} & D_{in} \end{bmatrix}$, $T_{out} = \begin{bmatrix} A_{out} & B_{out} \\ C_{out} & D_{out} \end{bmatrix}$ and $T_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$; $k = 1, 2, \ldots, n$. The variable n enumerates the number of repetition of a basic cell of the structure, which, in general, can consist of one or more periods. Then, assuming that all internal periods are identical, i.e., $T_1 = \cdots = T_n = T$, the $ABCD$ matrix of the overall structure can be represented as $\mathbf{T_n} = T_{in}(\prod^n T)T_{out}$. Letting

$$\mathbf{X_n} = T_{in}\prod^{n} T, \qquad \mathbf{T_n} = \mathbf{X_n}T_{out}. \tag{1}$$



(a)

(b)

Fig. 2. (a) Initial matrix; (b) matrix with an additional period.

Consequently

$$T_{out} = \mathbf{X_n^{-1} T_n}. \quad (2)$$

The $ABCD$ matrices of periodic structures with $n+1$ and $n+2$ repetitions of the basic cell can then be written as follows

$$\mathbf{T_{n+1}} = \mathbf{X_n} T T_{out} \quad (3)$$

$$\mathbf{T_{n+2}} = \mathbf{X_n} T T T_{out}. \quad (4)$$

Substituting (2) into (3), we get $\mathbf{T_{n+1}} = \mathbf{X_n} T \mathbf{X_n^{-1} T_n}$. Then, solving for T, we obtain

$$T = \mathbf{X_n^{-1} T_{n+1} T_n^{-1} X_n}. \quad (5)$$

Substituting (2) and (5) into (4) yields

$$\mathbf{T_{n+2}} = \mathbf{X_n X_n^{-1} T_{n+1} T_n^{-1} X_n X_n^{-1} T_{n+1} T_n^{-1} X_n X_n^{-1} T_n}.$$

Simplifying the above equation results in the recursive formula, $\mathbf{T_{n+2}} = \mathbf{T_{n+1} T_n^{-1} T_{n+1}}$. More generally,

$$\mathbf{T_{n+2m}} = \mathbf{T_{n+m} T_n^{-1} T_{n+m}}. \quad (6)$$

Equation (6) implies that the $ABCD$ matrix for a structure with $n+2m$ repetitions can be computed once the $ABCD$ matrices for the same structure with n and $n+m$ repetitions are known. The S-parameters of the structure can then be obtained from the $ABCD$ matrix.

Note the generality of the approach—typically, for closely spaced periods, a number of higher order modes would be excited at the discontinuities. Thus, the T matrices would be of dimensions $2K$ by $2K$, with K representing the number of modes necessary to fully characterize multi-mode relations in each section. By the same argument, T_{in} and T_{out} would be rectangular matrices with sizes $2I$ by $2K$ and $2K$ by $2I$, respectively. The matrices would relate the interaction between I modes from the input with K modes within the periodic structure. Fortunately, all internal higher order modal interactions are accounted for, but are not present in the final recursive formula. The multimode matrices T need not be explicitly known.

III. Results

To verify the proposed formula, a simple waveguide problem is considered. The simulated waveguide has the periodic characteristic shown in Fig. 1, with width and height of 25 mm × 5 mm and length of 595 mm along the axis of periodicity. Two experiments are performed: first, with the shaded boxes in Fig. 1 as solid perfect conductors and, second, with the boxes as dielectric gratings with relative permittivity of 5.6 and loss tangent of 0.004. The second waveguide contains more higher order modes than the first, and serves as a verification test that the formula applies to multimode propagation. The two ends of the waveguide are terminated in perfectly matched layers (eight layers, −80 dB, PML power 3.0 profile).

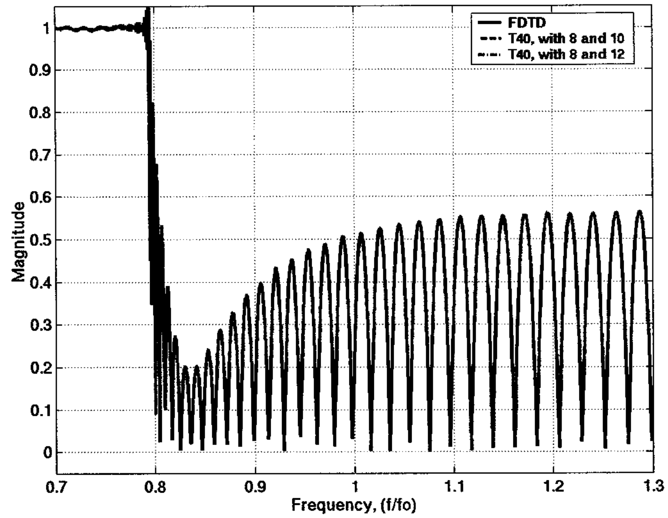


Fig. 3. S_{11} of rectangular waveguide with 40 perfect conductor periods.

In both cases, four FDTD simulations are performed with a) eight, b) ten, c) 12, and d) 40 periods. The structure is discretized with $\Delta x = \Delta y = \Delta z = 1$ mm, and is simulated for 20 000 time steps. The time interval between each time step, computed by the FDTD program, is 1.9 ps. S-parameters are computed from observations of the transverse electric field. S-parameters for cases a), b), and c), transformed to their respective $ABCD$ matrices, are denoted, $\mathbf{S_{a(8)}} \rightarrow \mathbf{T_{a(8)}}$, $\mathbf{S_{b(10)}} \rightarrow \mathbf{T_{b(10)}}$ and $\mathbf{S_{c(12)}} \rightarrow \mathbf{T_{c(12)}}$. The S-parameters for case d) are control values against which the results from the matrix equation will be compared.

$\mathbf{T_{a(8)}}$ and $\mathbf{T_{b(10)}}$ are used in the proposed formula, to predict the transmission matrix of a periodic structure with 40 periods from the knowledge of eight and ten in the following sequence of four iterations:

$$\mathbf{T_{12}} = \mathbf{T_{b(10)} T_{a(8)}^{-1} T_{b(10)}} \quad (7)$$

$$\mathbf{T_{16}} = \mathbf{T_{12} T_{a(8)}^{-1} T_{12}} \quad (8)$$

$$\mathbf{T_{24}} = \mathbf{T_{16} T_{a(8)}^{-1} T_{16}} \quad (9)$$

$$\mathbf{T_{40}} = \mathbf{T_{24} T_{a(8)}^{-1} T_{24}}. \quad (10)$$

The subscript "12," "16," "24," and "40" on the transmission matrices denote the number of periods that is being computed. The periodic increment "$\mathbf{m}$" in (7)–(10) is two, four, eight, and 16 periods, respectively. A similar procedure, except with only three iterations, is followed to obtain the transmission matrix of 40 periods from $\mathbf{T_{a(8)}}$ and $\mathbf{T_{c(12)}}$.

The results are transformed into S-parameters and plotted against the FDTD results for a periodic waveguide with 40 gratings. Since the structure is symmetric and reciprocal, $S_{21} = S_{12}$

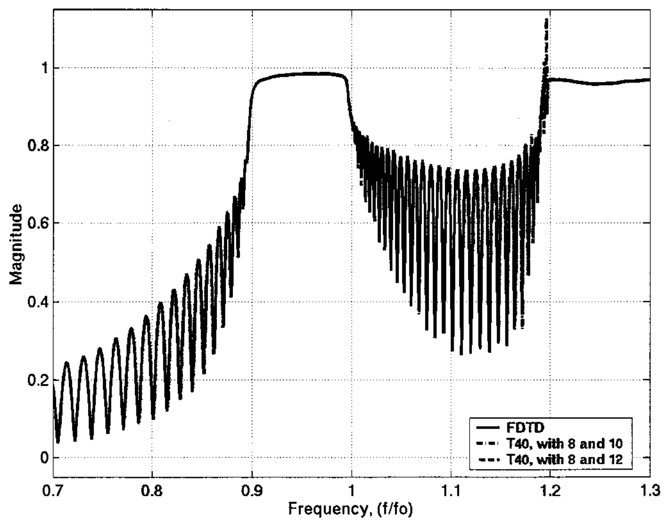


Fig. 4. S_{11} of rectangular waveguide with 40 dielectric periods.

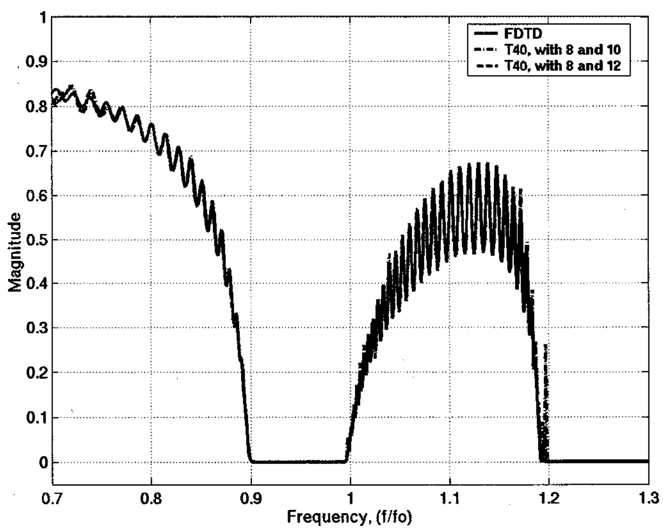


Fig. 5. S_{21} of rectangular waveguide with 40 dielectric periods.

and $S_{11} = S_{22}$. S_{11} of the waveguide with perfect conductor gratings is plotted in Fig. 3. S_{21} of the same structure exhibits similar accuracy and is not shown because of space constraint.

The results of the waveguide with dielectric gratings are plotted in Figs. 4 and 5. The frequency spectrum is normalized with respect to 12 GHz.

Figs. 3–5 show that the S-parameters calculated with the recursive formula have excellent agreement with that obtained from FDTD simulations. The utilization of the simple formula to compute the S-parameters of a structure with a large number of periodic perturbations is more time and computationally efficient than performing FDTD simulations on the same structure.

IV. Conclusion

Full wave electromagnetic simulators, such as FDTD, are versatile methods to extract the propagation characteristic of electromagnetic waves in arbitrary structures. They are, however, computationally intensive, and as the simulation space increases in size, the computation time becomes prohibitive. The proposed formula is an efficient tool to extend the results obtained from small-sized computations to structures with a large number of periodic gratings. The results are in excellent agreement with those obtained directly from FDTD.

References

[1] A. C. Cangellaris, M. Gribbons, and G. Sohos, "A hybrid spectral/FDTD method for the electromagnetic analysis of guided waves in periodic structures," *IEEE Microwave Guided Wave Lett.*, pp. 375–377, Oct. 1993.

[2] A. A. Shittu, S. Al-Bader, and H. Jamid, "Mode reflectivity of finitely-periodic dielectric waveguides," *Opt. Commun.*, vol. 114, pp. 242–246, 1995.

[3] P. Dansas and N. Paraire, "Fast modeling of photonic bandgap structures by use of a diffraction-grating approach," *J. Opt. Soc. Amer. A*, vol. 15, pp. 1586–1598, June 1998.

[4] G. V. Morozov, R. G. Maev, and G. W. F. Drake, "Exact analytic solution of the problem of reflection of an electromagnetic wave from a two-layer periodic dielectric structure," *Quantum Electron.*, vol. 28, pp. 977–982, 1998.

[5] D. T. McGrath and V. P. Pyati, "Periodic structure analysis using a hybrid finite element method," *Radio Sci.*, vol. 31, pp. 1173–1179, Sept.–Oct. 1996.

[6] W.-J. Tsay and D. M. Pozar, "Application of the FDTD technique to periodic problems in scattering and radiation," *IEEE Microwave Guided Wave Lett.*, vol. 3, pp. 250–252, Aug. 1993.

[7] P. Harms, R. Mittra, and W. Ko, "Implementation of the periodic boundary condition in the finite-difference time-domain algorithm for FSS structures," *IEEE Trans. Antennas Propagat.*, vol. 42, pp. 1317–1324, Sept. 1994.

[8] D. M. Pozar, *Microwave Engineering*. New York: Wiley, 1998, p. 211.